## 3.4 Simplification of Logical Expressions

With the calculation rules and logical equivalences presented so far, we can simplify propositional logical expressions by replacing variables contained therein consistently, i.e., by identical expressions. This is a proven tool to make complex formulas easier to understand without changing their truth content. This is called the substitution principle .

Please note that in the following examples we will use the abbreviated notations for conjunction and negation for a better overview in the complex examples. To help you get used to both types of notation, we will continue to express the somewhat simpler examples in detailed notation. 1. ¬P)Let P be a propositional expression. Then the formula P∨ P ∨ 0 ≡ PP ∨ ¬P ≡ 1(P ∨ 0) ∧ (P ∨ ¬P)(P ∨ (P ∨0) Examples: Simplification of expressions with the substitution principle

simplified as follows:

Because of the neutrality of , it follows that in the above formula with without changing the truth content of the formula.

Furthermore, . Thus we can replace

This results in . continues to apply and we can The formula can thus be simplified as follows:P∨ ∨ ¬P

P ∧ 1 with .

Let P and Q be logical expressions. Then the formula P ∧ (P ∨ Q) ∨ ¬P can be sim-

plified as follows:

apply the absorption rule regarding the conjunction to Because the conjunction has a higher binding strength than the disjunction, we canThus we can simplify the formula as follows:P ∨ ¬P. Thus we can replace is a tautology, it follows that P ∧ (P ∨ Q)P ∨ ¬P ≡ 1 with PP ∧ (P ∨ Q) and obtain . P ∨ ¬P and obtain . P ∧

P∧ P∨Q ∨ ¬P

(P ∨ (P ∧ Q

With the rule of absorption regarding conjunction with negation it follows that ≡ P ∧ (¬P ∨ Q)can be simplified as follows:With the rule of absorption regarding the disjunction it follows that Let Overall, we can simplify the formula as follows:P. With this we can simplify the formula in a first step to P and Q be logical expressions. Then the formula . . 1P ¬P (P ∨ (P ∧ QP ∨ (P ∧ Q) ≡)) ∧ (¬P ∨ Q ∧ (¬P ∨ QP ∧))

is as follows:Let P, Q, R, and P S be logical expressions. Then the formulaPPQQRS ∨ ¬PPQRSQ∨ PPPQRSQ¬P Q

Because the commutative law and the associative law are valid for the conjunction, we can exchange and parenthesize the order of the operands within parentheses at

By applying the law of distribution we can extract will. Thus we can transform the formula intoPR QS ∨ PR QS ∨ PR (QSPR) from the individual terms and

We can apply the distributive law once again and use the right partial expressionobtain PR QS ∨ QS ∨ QS

and reshape into QS ∨ QS ∨ QS

Q S∨S ∨ QS

is a tautology, the following applies:Because S∨S

This simplifies S∨S ≡ 1

to Q S∨S ∨ QS

With the identity of the conjunction it applies thatQ∧1∨ QS

And thus we can further simplify the formula∧ ≡

toBy slightly modifying the absorption rules regarding disjunction with negation, weQ∨ QS

simplifyto Q∨ QS (Please make it clear to yourself with the help of a truth table that this logical equivaQ∨S lence applies!)

We insert this simplified partial expression into the original formula and obtainPR Q∨S Overall, the formula can therefore be simplified as follows:By applying the distributive law again, the formula can finally be transformed intoPRQ ∨ PRS

PQRS ∨ PQRS ∨ PQRS ≡≡≡≡ PRPRPRPR QS∨ QS QQSS ∨ PR QS

≡≡